Europ. J. Combinatorics (1999) **20**, 75–80
Article No. eujc.1998.0270
Available online at http://www.idealibrary.com on IDEAL

A Characterization of Half-Dual Polar Graphs†

TUNG-SHAN FU

The half-dual polar graphs are characterized among distance-regular graphs with all singular lines of constant size of at least three by their parameters and some extra conditions.

1. INTRODUCTION

By a graph Γ, we shall mean a finite, simple and undirected graph. For x in $V(\Gamma)$, the vertex set of Γ, let $\Gamma_i(x)$ be the set of vertices of Γ at a distance i from x. A *distance-regular graph* of diameter d is one for which the parameters $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$, $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$ and $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$ depend only on the distance i between x and y. It is clear that $a_i = b_0 - b_i - c_i$. The sequence $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$ is called the *intersection array* of Γ. A family of distance-regular graphs, called *half-dual polar graphs*, derived from the *half-spin geometries* (defined below), are considered in this paper.

Let W be a vector space of dimension $2n$ over the finite field GF(q) of q elements with a quadratic form of Witt index n. It is well known that all maximal totally isotropic subspaces of W are of dimension n, and they can be partitioned into two classes with the property that two maximal totally isotropic subspaces A and B of W belong to the same class if and only if $A \cap B$ has even codimension in A (or equivalently, B). The *half-spin geometry* $D_{n,n}(q)$ is the point–line geometry $(\mathcal{P}, \mathcal{L})$ where the point set $\mathcal{P}$ is one of the two classes of maximal totally isotropic subspaces of W, and the line set $\mathcal{L}$ is the set of all totally isotropic subspaces of W of dimension $n-2$. See [1, 4] for more details. The collinearity graph of the half-spin geometry $D_{n,n}(q)$ is called the *half-dual polar graph*. The notation $D_{n,n}(q)$ is abusively used to denote the half-spin geometry as well as the half-dual polar graph. Therefore, the vertex set of $D_{n,n}(q)$ is one class of the maximal totally isotropic subspaces of W, and two vertices x and y of $D_{n,n}(q)$are adjacent if and only if $\dim(x \cap y) = n - 2$. The half-dual polar graph $D_{n,n}(q)$ is a distance-regular graph of diameter $d = \lfloor n/2 \rfloor$ with intersection array given by

$$b_i = q^{4i+1}\begin{bmatrix} n-2i \\ 2 \end{bmatrix}, \qquad c_i = \begin{bmatrix} 2i \\ 2 \end{bmatrix} \qquad (0 \le i \le d),$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the Gaussian coefficient with basis q. The singular lines (defined in Section 2) of $D_{n,n}(q)$ have size $q+1$, and the eigenvalues of $D_{n,n}(q)$ are given by

$$\theta_i = q^{2i+1}\begin{bmatrix} n-2i \\ 2 \end{bmatrix} - \frac{q^{2i}-1}{q^2-1} \qquad (0 \le i \le d).$$

Refer to [1] for more information.

The half-spin geometry was first characterized by Cooperstein in terms of points and lines, and then was included in a specific class of locally truncated diagram geometry by Cohen and Cooperstein [3]. Finally, a neat characterization of the half-spin geometry under conditions of points and singular subspaces alone was given by Shult [8].

†This research was partially supported by the National Science Council, Republic of China, Taiwan, under grant NSC 87-2115-M251-001.

0195-6698/99/010075 + 06 $30.00/0

The half-dual polar graphs $D_{n,n}(q)$ cannot be characterized simply by their intersection arrays because there is another family of distance-regular graphs [1, Theorem 9.4.10] sharing the same intersection array set, the singular lines of which have two different sizes. The main purpose of this paper, Theorem 3.1, is to characterize the half-dual polar graphs among distance-regular graphs with all singular lines of constant size by parameters and some geometric conditions. An incidence structure is first associated with the given distance-regular graph in a natural way and it is then shown to be the half-spin geometry following Shult's characterization in [8]. Some geometric backgrounds and the way we associate incidence structures with graphs are given in Section 2. The geometric properties of the half-dual polar graph are retrieved in Section 3, which pave the way to a proof of Theorem 3.1.

2. Geometric Preliminary

Some necessary background on incidence structure is given in the following for completeness.

An incidence structure $(\mathcal{X}, \mathcal{L})$ is called *semilinear* (or *partial linear*) if any two points are on at most one line, where $\mathcal{X}$ is the point set and $\mathcal{L}$ is the line set. Two points are *collinear* if they are on some line. The *collinearity graph* of $(\mathcal{X}, \mathcal{L})$ is the graph with vertex set $\mathcal{X}$ and edge set $\mathcal{E}$ consisting of pairs of collinear points of $\mathcal{X}$. A semilinear incidence structure $(\mathcal{X}, \mathcal{L})$ is called a *gamma space* if x is collinear with no, one, or all points of l for each pair $(x, l) \in (\mathcal{X}, \mathcal{L})$. A connected subset S of $\mathcal{X}$ is called a *subspace* if whenever a line l is incident with at least two points of S, then all points of l are in S. A subspace of $(\mathcal{X}, \mathcal{L})$ is called a *singular subspace* if any two of its points are collinear. The *singular rank* of a singular subspace S is the maximal integer n for which there exists a chain of singular subspaces $\emptyset \neq S_0 \subset S_1 \subset \cdots \subset S_n = S$. Here lines and planes are the singular subspaces of ranks 1 and 2, respectively. A *polar space* $(\mathcal{X}, \mathcal{L})$ is a gamma space satisfying the following '*one or all*' axiom: x is collinear with either one or all points of l for any point x and any line l not containing x. The rank of a polar space is defined to be the number $n + 1$ where n is its maximal singular rank. The local structure over the common neighborhood of two points at a distance two of the half-spin geometry is a nondegenerate polar space of rank 3. This structure plays an essential role in characterizing the half-spin geometries [3, 8] and the half-dual polar graphs [1, p. 278].

For a given graph Γ, we can associate it with an incidence structure naturally in the following way; let $x^{\perp} = \Gamma_1(x) \cup \{x\}$ for $x \in V(\Gamma)$, and let $S^{\perp} = \bigcap_{x \in S} x^{\perp}$ for $S \subseteq V(\Gamma)$ and $S^{\perp\perp} = (S^{\perp})^{\perp}$. In particular, $\{x, y\}^{\perp\perp}$ is called a *singular line* of Γ for any adjacent pair $x, y \in V(\Gamma)$. Let $\mathcal{L}$ be the collection of all singular lines of Γ. Clearly, singular lines are cliques and Γ is the collinearity graph of the incidence structure $(V(\Gamma), \mathcal{L})$. It is known [1, p. 29] that $(V(\Gamma), \mathcal{L})$ is semi-linear and turns out to be a gamma space with the property that each $l \in \mathcal{L}$ is in the intersection of all maximal cliques containing l if Γ is distance-regular. Here singular lines of distance-regular graphs are called *lines* for short.

Lemma 2.1. *Let Γ be a distance-regular graph. Then*

(1) *every maximal clique of Γ is a singular subspace,*
(2) *for any vertex x and any maximal clique M of Γ, $x^{\perp} \cap M$ is either empty or a singular subspace, and*
(3) *for any two vertices x, y of Γ at a distance two, $x^{\perp} \cap y^{\perp}$ is a gamma space.*

Proof. Let M be a maximal clique of Γ, and let l be a line of Γ such that $|l \cap M| \geq 2$. We want to show that $l \subseteq M$. Clearly, $M = \cap_{x \in M} x^{\perp}$. Since $(V(\Gamma), \mathcal{L})$ is a gamma space, $l \subseteq x^{\perp}$ for any point $x \in M$, and hence (1) follows. (2) is now evident.

To prove (3), let $w \in V(\Gamma)$ be a common neighbor of x and y, and let l be a line of Γ such that x, y and w are adjacent to two points $u, v \in l$. Then $x, y, w \in \{u, v\}^{\perp}$, and hence $l = \{u, v\}^{\perp\perp} \subseteq x^{\perp} \cap y^{\perp} \cap w^{\perp}$. (3) follows. □

Indeed, the above lemma holds in general for edge-regular graphs. Let C be a singular subspace of a graph Γ. Two vertices $u, v \in C^{\perp} - C$ are defined to be equivalent if $u^{\perp} = v^{\perp}$ in the subgraph induced on $C^{\perp} - C$; the equivalence class of u is denoted by $\bar{u}$. The *quotient graph* res(C) with respect to C is defined on the equivalence classes in $C^{\perp} - C$ with two classes $\bar{u}, \bar{v}$ adjacent if $\bar{u} \neq \bar{v}$ and $\bar{u} \cup \bar{v}$ is a clique. Refer to [1, Appendix A.8] for more details.

Let Γ be a distance-regular graph satisfying the conditions given in Theorem 3.1. Employing the technique of graph representations and the idea of quotient graphs with respect to singular subspaces of a graph [1, Propositions 3.7.4 and 4.4.2], we will retrieve some geometric properties of $(V(\Gamma), \mathcal{L})$ simply from the given parameters of Γ. We finally show that the associated incidence structure $(V(\Gamma), \mathcal{L})$ is the half-spin geometry and hence Γ is isomorphic to the half-dual polar graph $D_{n,n}(q)$.

3. The Main Theorem

The main result of this paper is given in Theorem 3.1.

THEOREM 3.1. *Suppose n, q and s are positive integers, and let Γ be a distance-regular graph of diameter $d = \lfloor n/2 \rfloor \geq 2$ with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ and second largest eigenvalue θ_1 such that $b_1 = q^5\left[{n-2 \atop 2}\right]$, $c_2 = (q^2+1)(q^2+q+1)$ and $\theta_1 = q^3\left[{n-2 \atop 2}\right] - 1$. Assume*

(A.1) *all lines of Γ have size $s + 1$, where $s \geq q \geq 2$;*
(A.2) *for any three pairwise adjacent vertices x, y; and z, $\{x, y, z\}^{\perp} \neq \{x, y, z\}^{\perp\perp}$; and*
(A.3) *for any vertex x and any maximal clique M, $|x^{\perp} \cap M| \neq 1, s + 1$.*

Then $s = q$ is a prime power and Γ is isomorphic to the half-dual polar graph $D_{n,n}(q)$.

For the rest of this paper, we assume that Γ is a given distance-regular graph satisfying the conditions mentioned above, and $(V(\Gamma), \mathcal{L})$ is the gamma space associated with Γ where $\mathcal{L}$ is the set of all lines of Γ. A triple x, y, z of pairwise adjacent vertices of Γ is called a *triangle* xyz if x, y and z are not contained in a line. Note that a triangle xyz is contained in some plane, and each plane contains at least $s^2 + s + 1$ points by (A.1) since all points on lines joining x with a point of the line through y and z belong to this plane.

Since $b_1/(\theta_1+1) = q^2 > 1$, by Proposition 4.4.2 [1], Γ has a weak $(q^4, q^2, 1)$-representation. Under conditions (A.1) and (A.2), Lemma 3.2 can be proved by applying Proposition 3.7.4 [1] together with the argument used in Theorem 3.7.5 [1]; its proof is included for completeness.

LEMMA 3.2.

(1) *Distinct maximal cliques of Γ intersect in a plane, a line, a point or the empty set.*
(2) *Every plane of Γ is contained in exactly two maximal cliques.*

PROOF. If (1) is not valid, then there exist maximal cliques intersecting in a singular subspace C properly containing a plane P. Denote by c the size of C and let $w \in C - P$. Then $c \geq s^3 + s^2 + s + 1$ since all points on lines joining w with a point of P belong to C. Clearly, the quotient graph res(C) contains at least two elements, and hence by (iii) of Proposition 3.7.4 [1], the following inequality holds.

$$sc(c - 1 - q^2) \leq (c - 1)^2 + q^4.$$

However $c \geq s^3 + s^2 + s + 1$ and $s \geq q \geq 2$ by (A.1),

$$\begin{aligned} sc(c-1-q^2) - (c-1)^2 - q^4 &= (c-1)(sc-c+1) - q^2sc - q^4 \\ &\geq (c-1)(sc-c+1) - s^3c - s^4 \\ &= (c-1)(sc-c+1-s^4) + s^3(sc-c-2s) \\ &> 0, \end{aligned}$$

a contradiction. Hence (1) follows.

To prove (2), let C be a plane. Then $c \geq s^2 + s + 1$, and the quotient graph $\mathrm{res}(C)$ is a coclique by (1). Let r denote the size of $\mathrm{res}(C)$. By (i) of Proposition 3.7.4 [1], $\mathrm{res}(C)$ has a $\left(\frac{q^2+sc}{1+(s-1)c}, 1, \frac{1-c}{q^2}\right)$ representation. In this representation, the condensed Gram matrix of the r-coclique is the 1×1 matrix with entry $\frac{1}{r}\left(\frac{q^2+sc}{1+(s-1)c} + (r-1)\frac{1-c}{q^2}\right)$. Since this must be a nonnegative number and $s \geq q \geq 2$, $(r-1)(c-1)(1+(s-1)c) \leq q^2(q^2+sc) \leq s^3(s+c)$, and hence

$$r \leq 1 + \frac{s+c}{c-1} = 2 + \frac{s+1}{c-1} < 3.$$

Therefore $r \leq 2$, i.e., C is contained in at most two maximal cliques. On the other hand, if C is contained in only one maximal clique, then $\{x, y, z\}^{\perp} = \{x, y, z\}^{\perp\perp}$ is the unique maximal clique containing C for any triangle xyz of C. This contradicts (A.2), and the assertion follows. □

The following is an immediate corollary under condition (A.3).

COROLLARY 3.3. *For any pair (x, M) of vertex x and maximal clique M not containing x, $x^{\perp} \cap M$ is either a plane or the empty set.*

Let xy denote the line through vertices x and y. According to Lemma 3.2, each triangle xyz determines a unique plane $\{x, y, z\}^{\perp\perp}$. For convenience, we let xyz denote the triangle as well as the plane $\{x, y, z\}^{\perp\perp}$. Let L_{xyz} and M_{xyz} be the two maximal cliques containing the plane xyz. Clearly, $xyz = L_{xyz} \cap M_{xyz}$ and $\{x, y, z\}^{\perp} = L_{xyz} \cup M_{xyz}$.

LEMMA 3.4. *If xyz is a plane, then no vertex in$L_{xyz} - M_{xyz}$ is adjacent to any vertex in $M_{xyz} - L_{xyz}$.*

PROOF. If $u \in L_{xyz} - M_{xyz}$ and $v \in M_{xyz} - L_{xyz}$ are adjacent, then $xyz \cup \{u, v\}$ is a clique containing the plane xyz but not contained in any of L_{xyz} and M_{xyz}, a contradiction. □

COROLLARY 3.5.

(1) *If L, M and N are three distinct maximal cliques containing a vertex x such that $L \cap M$ and $M \cap N$ are distinct lines, then $L \cap N$ is not a plane.*
(2) *For any line $xy \in \mathcal{L}$, there exist at least $s+1$ maximal cliques $L_1, L_2, \ldots, L_{s+1}$ such that $L_i \cap L_j = xy$, $1 \leq i, j \leq s+1$ and $i \neq j$.*

PROOF. If $L \cap N$ is a plane, then there exists a pair of adjacent vertices $u \in M \cap L - N$ and $v \in M \cap N - L$, contradicting Lemma 3.4, and hence (1) follows.

To prove (2), let M be a maximal clique with $M \cap xy = \{y\}$. By Corollary 3.3, there are at least $s+1$ distinct lines $yz_1, yz_2, \ldots, yz_{s+1}$ in the plane $x^{\perp} \cap M$ through y, and each line yz_i contributes a maximal clique L_{xyz_i} containing xy, $1 \leq i \leq s+1$. These maximal cliques intersect M in distinct lines, and hence (2) follows from (1). □

Let x, y be two vertices of Γ at a distance two. We shall show, in the following, that $x^\perp \cap y^\perp$ carries the structure of a polar space of rank 3. Let $\mathcal{M}$ denote the set of all maximal cliques M containing y such that $x^\perp \cap M$ is nonempty, and let $\mathcal{B} := \{x^\perp \cap M \mid M \in \mathcal{M}\}$. Members of $\mathcal{B}$ are called *blocks*. Clearly, each block is a plane of Γ as well as a maximal clique of the subgraph induced on $x^\perp \cap y^\perp$, and $\mathcal{B}$ exhausts all maximal cliques of $x^\perp \cap y^\perp$ by Lemma 3.2 and Corollary 3.3.

PROPOSITION 3.6.

(1) $s = q$.
(2) *Every block of* $x^\perp \cap y^\perp$ *is a projective plane of order* q.
(3) *For any block* C *of* $x^\perp \cap y^\perp$ *and any point* $w \in x^\perp \cap y^\perp - C$, $w^\perp \cap C$ *is a line.*

PROOF. For a given block $B \in \mathcal{B}$, let M be the maximal clique containing y such that $x^\perp \cap M = B$. Define

$$\mathcal{B}_M := \{\, x^\perp \cap L \mid L \in \mathcal{M} \text{ and } L \cap M \text{ is a line}\}.$$

If $C \in \mathcal{B}_M$ is a block intersecting B in a line uv, then $C = x^\perp \cap L$ for some maximal clique L containing y, and hence $L \cap M$ contains the plane yuv, a contradiction. Moreover, if $C_1, C_2 \in \mathcal{B}_M$ are two distinct blocks intersecting in a line uv, then $C_i = x^\perp \cap L_i$ for some maximal clique L_i containing y and the line $L_i \cap M$, $i = 1, 2$, and hence $L_1 \cap L_2$ contains the plane yuv, contradicting Corollary 3.5(1). Hence distinct blocks of $\mathcal{B}_M \cup \{B\}$ have at most one point in common. Furthermore, let

$$\Gamma_1(B) := \{\, w \in x^\perp \cap y^\perp \mid w \notin B \text{ and } w^\perp \cap B \text{ is nonempty}\},$$

and

$$\mathcal{R} := \{(w, C) \mid w \in \Gamma_1(B), C \in \mathcal{B}_M, w \in C \text{ and } C \cap B \neq \emptyset\}.$$

We now count the set $\mathcal{R}$ in two ways. Since $x^\perp \cap y^\perp$ is a gamma subspace, $w^\perp \cap B$ is a line or a single point in $x^\perp \cap y^\perp$ for any $w \in \Gamma_1(B)$. Therefore, each point $w \in \Gamma_1(B)$ is in at most $s + 1$ blocks C of $\mathcal{B}_M$ intersecting B. On the other hand, each point $z \in B$ is in at least s blocks C of $\mathcal{B}_M$ by Corollary 3.5(2), and each such block contributes at least $s^2 + s$ points w to $\mathcal{R}$. It follows that $|\Gamma_1(B)|(s + 1) \geq |\mathcal{R}| \geq |B|s(s^2 + s)$, and hence $|\Gamma_1(B)| \geq |B|s^2$. Since $s \geq q$ and the disjoint union of B and $\Gamma_1(B)$ is contained in $x^\perp \cap y^\perp$, we have

$$\begin{aligned}
(q^2+1)(q^2+q+1) &= |x^\perp \cap y^\perp| \qquad \text{(i.e., } c_2\text{)}\\
&\geq |B| + |\Gamma_1(B)|\\
&\geq |B|(s^2+1)\\
&\geq (s^2+s+1)(s^2+1)\\
&\geq (q^2+q+1)(q^2+1).
\end{aligned}$$

Hence $s = q$, $|B| = q^2 + q + 1$ and $x^\perp \cap y^\perp = B \cup \Gamma_1(B)$. Moreover, B is a projective plane of order q and $x^\perp \cap y^\perp$ is connected. Note that w is in exactly $q + 1$ blocks of $\mathcal{B}_M$ intersecting B for any point $w \in \Gamma_1(B)$, and each such block consists of $q^2 + q + 1$ points. Since B is an arbitrary block of $x^\perp \cap y^\perp$, (2) and (3) follow immediately. □

PROPOSITION 3.7.

(1) *Every maximal clique of* Γ *is a projective space of order* q, *and hence* q *is a prime power.*

(2) *For any two vertices* $x, y \in V(\Gamma)$ *at a distance two,* $x^\perp \cap y^\perp$ *is a polar space of rank 3.*

PROOF. If x, y and z are distinct vertices of a maximal clique M not contained in any line, then x, y and z belong to a projective plane xyz of order q by Lemma 3.2 and Proposition 3.6. Hence M is a projective space of order q and q is a prime power by the classical fundamental theorem of finite geometry of Veblen and Young [10].

To prove (2), we shall show that $x^\perp \cap y^\perp$ satisfies the 'one or all' axiom. If uv is a line of $x^\perp \cap y^\perp$, then there exists a block $B = x^\perp \cap L_{yuv} \subseteq x^\perp \cap y^\perp$ containing this line. For any vertex $w \in x^\perp \cap y^\perp$, if $w \in B$ then $uv \subseteq w^\perp$; otherwise, $w \notin B$, in which case either $w^\perp \cap B = uv$, or $w^\perp \cap B$ and uv are two distinct lines of the projective plane B by Proposition 3.6. Hence, either $uv \subseteq w^\perp$ or $w^\perp \cap uv$ is a single point in $x^\perp \cap y^\perp$. Since maximal singular subspaces of $x^\perp \cap y^\perp$ are projective planes, we conclude that $x^\perp \cap y^\perp$ is a polar space of rank 3. □

We can now prove our main result.

*Proof of Theorem 3.1.*By Corollary 3.3 and Proposition 3.7, the gamma space $(V(\Gamma), \mathcal{L})$ associated with Γ is a parapolar space with thick lines satisfying hypothesis (**) of the main theorem in [8]. In particular, $x^\perp \cap M$ is a projective plane for any pair (x, M) of vertex x and maximal clique$M$ not containing x. With the further property that every plane is contained in exactly two maximal cliques, we conclude that $(V(\Gamma), \mathcal{L})$ is a half-spin geometry $D_{n,n}(q)$ or a polar space of type D_4 [9, 7.12 and 8.4.3]. Hence the graph Γ, the collinearity graph of $(V(\Gamma), \mathcal{L})$, is isomorphic to the half-dual polar graph $D_{n,n}(q)$. □

REFERENCES

1. A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer, Berlin, 1989.
2. A. M. Cohen, On a theorem of Cooperstein, *Europ. J. Combinatorics*, **4** (1983), 107–126.
3. A. M. Cohen and B. N. Cooperstein, A characterization of some geometries of Lie type, *Geom. Ded.*, **15** (1983), 73–105.
4. B. N. Cooperstein, A characterization of some Lie incidence structures, *Geom. Ded.*, **6** (1977), 205–258.
5. T.-S. Fu and T. Huang, A unified approach to a characterization of Grassmann graphs and bilinear forms graphs, *Europ. J. Combinatorics*, **15** (1994), 363–373.
6. T.-S. Fu and T. Huang, A characterization of two classes of strongly regular graphs, *J. Geometry*, **52** (1995), 91–100.
7. E. Shult, Characterizations of Lie incidence geometries, in: *Surveys in Combinatorics*, E. Keith Lloyd (ed.), *London Math Soc. Lecture Note*, **82** (1982), 157–184. Cambridge University Press, Cambridge.
8. E. Shult, A remark on Grassmann spaces and half-spin geometries, *Europ. J. Combinatorics*, **15** (1994), 47–52.
9. J. Tits, *Buildings of Spherical Type and Finite BN-pairs*, *Lecture Notes in Mathematics*, **386** (1974), Springer, Berlin.
10. O. Veblen and J. W. Young, *Projective Geometry*, Ginn, Boston, 1916.

Received 25 February 1998 and accepted 12 October 1998

T.-S. FU
National Ping-Tung Institute of Commerce,
Ping-Tung 900,
Republic of China, Taiwan
E-mail: `tsfu@sun1.nptic.edu.tw`